Signal Recovery and Amplitude-Phase Tracking in Wireless Communications

Dawei Huang & Lang White

Abstract— **This paper addresses the problem of demodulating a digital communications signal in the presence of unknown channel complex gain. We derive the Cramer-Rao lower bound for tracking the channel gain in the presence of additive white Gaussian noise. This bound establishes effective lower bounds for signal-to-noise ratio on the channel, given the statistics of the channel gain variations. Several approaches for tracking the gain are derived based on the application of Bayes theorem, and the approximation of soft decision symbol statistics by a Gaussian density. The performance of the tracker is examined using simulated and real data.**

I. INTRODUCTION

The problem of recovering a transmitted digital data sequence in the presence of unknown complex channel gain is fundamental in digital communications systems and is, indeed, a basic issue in synchronising the receiver to the transmitter. Gain and phase variations on the channel may be due to the effects of the propagation medium, or due to mismatches existing between the receiver and transmitter. Sometimes the gain is unknown but constant, whilst on other occasions it may be time varying, necessitating tracking of the gain variations.

The standard approach is to utilise a combined automatic gain control - phase-locked loop algorithm to estimate the channel gain and phase. This approach can be shown to arise as an approximation to the extented Kalman filter (EKF) applied to a random walk model of the amplitude and frequency offset processes [1]. Because the EKF is derived by linearisation, its performance can degrade rapidly in the presence of noise and decision errors. It is the objective of this work to design better tracking algorithms, and to examine the fundamental limitation in terms of SNR imposed on the bit-error rate (BER) performance of the system due to channel gain variations. The paper focusses primarily on PSK modulation, however in many cases, the tracker can also be applied to QAM data.

The layout of the paper is as follows : firstly we derive our signal model and thence the Cramer-Rao bound (CRB) on tracking performance in the presence of the underlying digital modulation. We then examine separately the issues of amplitude and phase tracking comparing the tracking performance to the CRB. We include both a decision directed (hard decision) phase tracker and a non-decision directed (Bayesian) method. Finally, we develop a Bayesian approach for soft decision symbol estimation and tracking. Performance is examined using simulations and real TDMA data.

II. SIGNAL MODEL AND CRB

Let

$$\begin{aligned} \omega_{t+1} &= \omega_t + u_t, \quad \phi_{t+1} = \phi_t + \omega_t, \\ A_{t+1} &= A_t + v_t, \\ z_t &= |A_t| \exp\{i(\phi_t + \psi_t)\} + n_t, \end{aligned} \tag{1}$$

where $u_t \sim iid\,N(0,\sigma_u^2)$, $v_t \sim iid\,N(0,\sigma_v^2)$ and $n_t \sim iid\,N(0,\sigma^2)$. The signal may also not be present at some times (ie A_t could be zero on some intervals). The process $\{\psi_t\}$ is iid uniformly distributed on the discrete set $\{2k\pi/M\}, k = 0,...,(M-1)$. The objective is to recover the sequence ψ_t from the measurements $z_0, \ldots, z_t$. Clearly this involves tracking ϕ_t to adapt the decision device. Typically the decision device is given by

$$\hat{\psi}_t = Q\left(z_t e^{-i\hat{\phi}_{t|t-1}}\right), \tag{2}$$

where Q is a quantisation based on the nearest neighbour in the above discrete set and $\hat{\phi}_{t|t-1}$ is a one step prediction for the ϕ_t process. The frequency tracker then uses $\hat{\psi}_t$ in the above model to update its estimates. This is known as (hard) decision direction.

In eqn (1), let $x_t = [\omega_t, \phi_t, A_t]'$, $n_{t,1}$ and $n_{t,2}$ be the real and imaginary parts of n_t, we can show that

$$\frac{\partial \log p(z_t|x_t)}{\partial x_t} = \begin{bmatrix} 0 \\ -|A_t| \frac{n_{t,1}\sin(\phi_t+\psi_t) - n_{t,2}\cos(\phi_t+\psi_t)}{\sigma^2} \\ \pm \frac{-n_{t,1}\cos(\phi_t+\psi_t) + n_{t,2}\sin(\phi_t+\psi_t)}{\sigma^2} \end{bmatrix}.$$

Let $\mathrm{Var}(\mathrm{A_t}) = \sigma^2_{\mathrm{A,t}}$, then we have

$$E\left[\frac{\partial \log p(z_t|x_t)}{\partial x_t}\frac{\partial \log p(z_t|x_t)}{\partial x_t'}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\sigma^2_{A,t}}{\sigma^2} & 0 \\ 0 & 0 & \frac{1}{\sigma^2} \end{bmatrix}.$$

Let B_t and b_t be the Cramer-Rao bound for $[\omega_t, \phi_t]$ and A_t respectively. Using the result in [2], we have

$$\begin{aligned} B_t = TB_{t-1}T' - \left(\begin{bmatrix} 1 & 1 \end{bmatrix} B_{t-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\sigma^2}{\sigma^2_{A,t}}\right)^{-1} \\ \times TB_{t-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} B_{t-1}T', \end{aligned} \tag{3}$$

$$b_t = b_{t-1} + \sigma_v^2 - \frac{\left(b_{t-1}+\sigma_v^2\right)^2}{\sigma^2 + b_{t-1} + \sigma_v^2} = \frac{\left(b_{t-1}+\sigma_v^2\right)\sigma^2}{\sigma^2 + b_{t-1} + \sigma_v^2}.$$

D. Huang is with the Department of Mathematics, Queensland University of Technology, Brisbane Australia. email : d.huang@fsc.qut.edu.au (corresponding author)

L. B. White is Head, Signals Analysis Discipline, Defence Science and Technology Organisation, Australia. email : Lang.White@dsto.defence.gov.au

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Thus,

$$b_t \to b = \frac{\sqrt{\sigma_v^2 + 4\sigma^2} - \sigma_v}{2}\sigma_v. \qquad (4)$$

Also, B_t tends to a limit matrix too, which can be calculated by the recursive formula (3), or by solution of the corresponding algebraic Riccati equation.

To obtain the formula for CRB, we only need to assume that A_t, n_t, ϕ_t and ψ_t are independent and that n_t is complex Gaussian $N\left(0, \sigma^2\right)$. It is clear that the distributions of ϕ_t and ψ_t may effect on the problem, they do not appear in the CRB. This is because that the CRB is less than or equal to the minimum of MSE over all possible distributions of ϕ_t and ψ_t. However, we really use the information about the distributions of ϕ_t and ψ_t in our tracking algorithms to get closer results to the CRB.

A direct application of CRB is to assess the lower bound on SNR for which the decision device (2) functions effectively. Here we assume that A_t takes a constant value A, and $SNR = 10 \times \log_{10}\left(\frac{A^2}{2\sigma^2}\right)$. According to [3] (we will investigate this result in Section 3), we have

$$\arg\left(z_t e^{-i\hat{\phi}_{t|t-1}}\right) = \psi_t + \phi_t - \hat{\phi}_{t|t-1} + \varepsilon_t. \qquad (5)$$

Using the CRB, with $H = [0, 1]$

$$E\left(\phi_t - \hat{\phi}_{t|t-1}\right)^2 \geq HTBT'H' = c^2,$$

then the variance of the error term in (5) is lower bounded by $c^2 + \sigma^2/A^2$ in variance. The following table gives the value of $3\sqrt{c^2 + \sigma^2/A^2}$:

SNR:	-5	0	5	10	15	20
$\sigma_u^2 = 10^{-4}$	4.02	2.31	1.33	0.78	0.46	0.28
$\sigma_u^2 = 10^{-5}$	3.90	2.22	1.27	0.73	0.42	0.25
$\sigma_u^2 = 10^{-6}$	3.85	2.18	1.24	0.70	0.40	0.23

Even if we choose $M = 2$, the distance we need to distinguish the two signals is $\pi/2 = 1.57$. Thus, to get error rate less than 1%, no matter what filtering method we use, the decision device (2) may only work for SNR≥5. The simulation results in (11) have confirmed this conclusion.

III. TRACKING FOR A_t

In this section, we derive an approximate Bayesian filter for tracking of the amplitude. Using polar co-ordinates, we have

$$z_t = |A_t| \exp i\left(\phi_t + \psi_t\right) + n_t = \rho_t e^{i\alpha_t}. \qquad (6)$$

Then, the density for the signal magnitude ρ_t is given by the Rician density [4]

$$p(\rho_t|A_t) = \frac{\rho_t}{\sigma^2}\exp\left(-\frac{\rho_t^2 + A_t^2}{2\sigma^2}\right) I_0\left(\frac{\rho_t A_t}{2\sigma^2}\right), \qquad (7)$$

where $I_0(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{(k!)^2}$ is the modified Bessel function of the first kind of zero order. Let $L_0(x) = \log I_0(x)$, then (7) can be written

$$p(\rho_t|A_t) \propto \exp\left[-\frac{\rho_t^2 - 2\sigma^2 L_0\left(\frac{\rho_t A_t}{2\sigma^2}\right) + A_t^2}{2\sigma^2}\right]. \qquad (8)$$

We now apply the Gaussian approximation idea from [2] to obtain the approximation (by linearisation about current predicted amplitude $\hat{A}_{t|t-1}$)

$$p(\rho_t|A_t) \approx \frac{\rho_t}{\sigma^2}\exp\left\{-\frac{\left[A_t - \rho_t l_0\left(\frac{\rho_t \hat{A}_{t|t-1}}{2\sigma^2}\right)\right]^2}{2\sigma^2}\right\}, \qquad (9)$$

where $l_0(x) = L_0'(x) = I_1(x)/I_0(x)$.

Then, using the Gaussian product lemma in [2] and Baysian formula, we derive the filter for tracking A_t :

$$s_t = \frac{\left(s_{t-1} + \sigma_v^2\right)\sigma^2}{\sigma^2 + s_{t-1} + \sigma_v^2} \to \frac{\sqrt{\sigma_v^2 + 4\sigma^2} - \sigma_v}{2}\sigma_v, \quad \text{as } t \to \infty,$$

$$\hat{A}_t = \hat{A}_{t-1} + \frac{s_t}{\sigma^2}\left[\rho_t l_0\left(\frac{\rho_t \hat{A}_{t|t-1}}{2\sigma^2}\right) - \hat{A}_{t-1}\right].$$

Simulation results: Let $\sigma^2 = 1$, the following table gives the sample Mean Squares Error which is taken by 100 repetitions and calculated by $MSE = \frac{1}{100}\sum_{i=1}^{100}\frac{1}{1000}\sum_{t=1000}^{2000}\left(\left|\hat{A}_t^{(i)}\right| - \left|A_t^{(i)}\right|\right)^2$:

σ_v^2	.00001	.0001	.001	.01	.1	1
CRB	0.0032	0.010	0.031	0.095	0.270	0.618
MSE	0.0038	0.013	0.047	0.113	0.284	0.628

IV. FILTERING FOR PHASE

In this section we consider A_t in (1) as a positive constant A. Let $\alpha_t = \arg(z_t)$, using model (1) with $A_{t+1} = A$ can be approximated by

$$\omega_{t+1} = \omega_t + u_t, \quad \phi_{t+1} = \phi_t + \omega_t,$$
$$\alpha_t + 2k_t\pi = \phi_t + \psi_t + \varepsilon_t. \qquad (10)$$

This is a linear model, the only problem is that we do not know the integer k_t. Let $\delta_t = \hat{\phi}_{t|t-1} - \alpha_t$, a simple method is to estimate k_t based on previous estimate :

$$k_t = \frac{\delta_t + \psi_t + \left(\phi_t - \hat{\phi}_{t|t-1}\right) + \varepsilon_t}{2\pi} \approx \frac{\delta_t + \psi_t}{2\pi}.$$

Let $[x]$ be the integer most close to x. When $\frac{\delta_t + \psi_t}{2\pi}$ is close to an integer, there is a large chance that $k_t = \left[\frac{\delta_t + \hat{\psi}_t}{2\pi}\right]$, especially if SNR is high. We choose $\hat{\psi}_t$ in $\left\{\frac{2k\pi}{M}, k = 0, 1..., M-1\right\}$ to make $\frac{\delta_t + \hat{\psi}_t}{2\pi}$ closest to an integer.

Thus, the standard Kalman filter gives us the following

$$S_{t+1|t} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} S_t \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & 0 \end{bmatrix},$$

$$S_{t+1} = S_{t+1|t} - \frac{1}{\sigma_\alpha^2 + S_{t+1|t}^{(2,2)}} S_{t+1|t} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} S_{t+1|t},$$

$$\hat{X}_{t+1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \hat{X}_t + \frac{2k_t\pi - \delta_t - \hat{\psi}_t}{\sigma_\alpha^2} S_{t+1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Simulation results: For (1) with $A_{t+1} = A$, $\{\psi_t\}$ being zero and an iid uniform binary sequence respectively, we tested the above method for 100 repetitions with each sample size 2000, here and after the MSE for phase tracking is measured moduli 2π :

σ_u^2	$SNR:$	-5	0	5	10	15
10^{-5}	$C-R$	0.108	0.045	0.019	0.0077	0.0032
$\psi_t = 0$	MSE	0.360	0.073	0.023	0.0081	0.0032
Binary	MSE	3.114	2.588	0.309	0.0085	0.0032
10^{-6}	$C-R$	0.062	0.026	0.011	0.0045	0.0019
$\psi_t = 0$	MSE	0.114	0.041	0.014	0.0047	0.0019
Binary	MSE	3.034	1.593	0.057	0.0050	0.0020

When $\{\psi_t\}$ is an iid uniform binary sequence, the percentage BERs are listed in the following table :

$\sigma_u^2 \backslash SNR:$	-1	0	1	2	3	4
10^{-5}	41.61	33.43	21.80	10.93	3.80	3.36
10^{-6}	17.26	14.92	8.13	5.81	3.36	1.33
$\sigma_u^2 \backslash SNR:$	5	6	7	8	9	10
10^{-5}	1.63	0.25	0.07	0.03	0.002	0
10^{-6}	0.60	0.12	0.07	0.02	0.003	0

(11)

A. Use of filtered phase estimates

In the decision device (2), one-step predictions of the phase process is used in order to gain a causal structure, since the filtered estimates depend on the symbol at that time. An alternative approach is based on the use of filtered estimates $\hat{\phi}_t$ directly, ie one which does not depend on ψ_t. Then a new decision device can be derived as

$$\hat{\psi}_t = Q\left(z_t e^{-i\hat{\phi}_t}\right). \tag{12}$$

To use this decision device, we need to estimate $\hat{\phi}_t$ without knowledge of ψ_t. By marginalising over the data symbols assuming uniform *a priori* probabilities we have that

$$p\left(z_t|A_t, \phi_t\right) = \frac{1}{2M\pi\sigma^2} \sum_{j=0}^{M-1} \exp\left[-\frac{\rho_t^2 - 2\rho_t A_t \cos\left(\phi_t + 2j\pi/M - \alpha_t\right) + A_t^2}{2\sigma^2}\right].$$

We use method in Section 3 to obtain $\left\{\hat{A}_t\right\}$ and let

$$C_t = \frac{1}{2M\pi\sigma^2} \exp\left[-\frac{\left(\rho_t - A_t\right)^2}{2\sigma^2}\right], \quad \sigma_t^2 = \frac{\sigma^2}{\rho_t \hat{A}_t},$$

using the second order Taylor expansion for $cos\,(\phi_t + 2k\pi/M -$

$$p\left(z_t|x_t\right) \approx C_t \sum_{j=0}^{M-1} \exp\left[-\frac{\left(\phi_t + 2j\pi/M - \alpha_t - 2k_{t,j}\pi\right)^2}{2\sigma_t^2}\right]. \tag{13}$$

This leads to a Gaussian mixture (Gaussian sum) model. Substituting (13) into the Baysian formula

$$p\left(x_t|Z_{t-1}\right) = \int p\left(x_t|x_{t-1}\right) p\left(x_{t-1}|Z_{t-1}\right) dx_{t-1},$$

$$p\left(x_t|Z_t\right) = \frac{p\left(z_t|x_t\right) p\left(x_t|Z_{t-1}\right)}{\int p\left(z_t|x_t\right) p\left(x_t|Z_{t-1}\right) dx_t},$$

and using the Gaussian product lemma in [2], theoretically we can calculate the posterior pdf $p\left(x_t|Z_t\right)$. However, the number of Gaussian components in it will increases exponentially and practically it does not work. A tractable approach is the following Maximum Entropy method [2].

Assume that we have pdf $p\left(x_{t-1}|Z_{t-1}\right) = G\left(x_{t-1}; \hat{x}_{t-1}, S_{t-1}\right)$ here and after G denotes a Gaussian pdf. Since for pdfs sharing the same first and second moments, the Gaussian pdf has the maximal entropy, we only need to use the mean and variance of $p\left(z_t|x_t\right) p\left(x_t|Z_{t-1}\right) / \int p\left(z_t|x_t\right) p\left(x_t|Z_{t-1}\right) dx$ to construct a Gaussian pdf for $p\left(x_t|Z_t\right)$. Let

$$\hat{\phi}_{t|t-1} = [1,1] \times \hat{x}_{t-1}, \quad k_{t,j} = \left[\frac{\hat{\phi}_{t|t-1} + j\pi/M - \alpha_t}{2\pi}\right],$$

$$\alpha_{t,j} = \alpha_t + 2\pi k_{t,j} - j\pi/M, \quad d_{t,j} = \alpha_{t,j} - \hat{\phi}_{t|t-1},$$

$$K_t = \frac{1}{\sigma_t^2 + HS_{t|t-1}H'} S_{t|t-1}H',$$

$$w_{t,j} = \frac{G\left(\hat{\phi}_{t|t-1}; \alpha_{t,j}, \sigma_t^2 + HS_{t|t-1}H'\right)}{\sum_{j=0}^{M-1} G\left(\hat{\phi}_{t|t-1}; \alpha_{t,j}, \sigma_t^2 + HS_{t|t-1}H'\right)},$$

$$S_t' = S_{t|t-1} - K_t H S_{t|t-1}. \tag{14}$$

Then the maximal entropy pdf [2] is:

$$p\left(x_t|Z_t\right) = G\left(x_t, \hat{x}_t, S_t\right),$$

$$\hat{x}_t = \hat{x}_{t|t-1} + K_t \sum_{j=0}^{M-1} w_{t,j} d_{t,j},$$

$$S_t = S_t' + \sum_{j=0}^{M-1} w_{t,j} \left(\sum_{n=0, n\neq j}^{M-1} w_{t,j} d_{t,j}\right)^2 K_t K_t'.$$

Simulation result: For (1) with $A_t = A$ and $\{\psi_t\}$ being an iid binary sequence, we tested the above method for 100 repetitions with each sample size 2000. The percentage BERs are listed in the following table:

$\sigma_u^2 \backslash SNR:$	-1	0	1	2	3	4
10^{-5}	26.59	17.63	8.14	6.86	2.43	1.37
10^{-6}	12.55	13.39	7.80	5.77	2.42	3.21
$\sigma_u^2 \backslash SNR:$	5	6	7	8	9	10
10^{-5}	0.64	0.25	0.08	0.024	0.0035	0.0005
10^{-6}	0.62	0.24	0.08	0.017	0.0035	0.0005

Comparing with (11), we can see the error rates reduce significantly for low SNR.

V. A SOFT DECISION PHASE TRACKER

The above maximum entropy approximation procedure can also be applied using *a posteriori* statistics of the transmitted symbols rather than *a priori* (uniform) statisitcs. Recall that the purpose of using the *a priori* statistics in the preceeding section was to allow the calculation of of the (more accurate) filtered phase estimates independently of the symbol value. In this case, we still compute predictions, but on the basis of symbol likelihoods (soft decisions). The same formulation is used, but the soft decision likelihoods are given by

$$\begin{aligned}\mu_t(j) &= \Pr\{\psi_t = 2j\pi/M | Z_t^-, \hat{\phi}_{t|t-1}\} \\ &= c_t \exp\{A\rho_t \cos(\alpha_t - \hat{\phi}_{t|t-1} - 2j\pi/M)/\sigma^2\}.\end{aligned}$$

where c_t normalises the sum of the $\mu_t(j)$ to unity. These weighting factors are then included in the Gaussian sum approximation above in equation (14). More details are given in [5].

VI. Results

In addition to the various simulation results listed above, we compared a number of phase-amplitude trackers on some real data. The data used was TDMA data with QPSK (M=4) modulation. The acquisition problem is very difficult in this case, not primarily because of channel variations associated with any particular user, but because of the number of different users each having different gains and frequency offsets. The data is also NOT differentially coded so that phase jumps of multiples of $\pi/2$ are not permitted. The following table shows the number of frames out of a total of 100 successfully demodulated using the following trackers : (i) Decision direction with fixed filter, (ii) EKF, (iii) Decision directed method of this paper (DDHW) and (iv) soft decision method of this paper (SDHW). The number in brackets shows the number demodulated with BER < 0.2 but with phase jumps possibly present, whilst the first number in each case shows the number successfully demodulated without phase jumps. These results clearly illustrate the performance of the method in terms of tracking stability.

Slot	Fixed	DDEKF	DDHW	SDHW
1	21 (98)	24 (91)	17 (97)	23 (94)
2	**47** (100)	78 (95)	**87** (96)	95 (98)
3	**5** (100)	99 (100)	**100** (100)	96 (97)
4	**75** (100)	98 (100)	**99** (100)	97 (99)
5	**48** (96)	100 (100)	**99** (99)	97 (97)
6	**2** (100)	100 (100)	**100** (100)	99 (99)
7	10 (54)	16 (53)	15 (55)	15 (54)
8	**45** (99)	83 (97)	**88** (99)	95 (100)

Table 1 - Performance of tracker algorithms

VII. Conclusion

This paper has examined a number of Bayesian approaches to amplitude-phase tracking in digital communications. We have derived Cramer-Rao bounds for tracking in the presence of modulation and compared these bounds to the performance of the various trackers. Both decision directed and non decision directed techniques are considered. We have concluded that the Bayesian non-decision directed method is superior at low SNR, but as SNR improves, all the techniques considered here approach optimal performance.

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